

The phase of the scattering operator from the geometry of certain infinite-dimensional groups

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Abstract

We revisit the computation of the phase of the Dirac fermion scattering operator in external gauge fields. The computation is through a parallel transport along the path of time evolution operators. The novelty of the present paper compared with the earlier geometric approach by Langmann and Mickelsson [LM] is that we can avoid the somewhat arbitrary choice in the regularization of the time evolution for intermediate times using a natural choice of the connection form on the space of appropriate unitary operators.

1 Introduction

In quantum mechanics the scattering operator S for Dirac fermions in an external vector potential A is computed from the asymptotics of the unitary time evolution operator $U(t)$,

$$i\frac{\partial}{\partial t}U(t) = D_A U(t)$$

where D_A is the Hamilton operator and the initial condition is $U(0) = 1$, assuming that the potential A is switched off for times $t < 0$. The scattering operator is then $S = \lim_{t \rightarrow \infty} U_0(t)^{-1}U(t)$ and this limit exists (in strong operator topology) when the potential is smooth and goes to zero enough rapidly at infinity. Here $U_0(t)$ is the free time evolution corresponding to $A = 0$ with the initial condition $U_0(0) = 1$.

In quantum field theory one needs to promote S to an unitary operator \hat{S} in the fermionic Fock space. According to the Shale-Stinespring [ShSt] theorem this is possible when the off-diagonal blocks of S in the energy polarization (with respect to the free Dirac hamiltonian D_0) are Hilbert-Schmidt, and this is indeed the case under the above mentioned restrictions on the potential A . The unitary operators with this restriction for a *restricted unitary group*, denoted by U_{res} . This groups and its representation theory was studied in detail in [PrSe]. The only problem is that the phase of \hat{S} is not determined by the canonical quantization procedure.

The physics question is: Why bother about the phase? Physicists know how to handle these things in perturbation theory since the work of Richard Feynman, Julian Schwinger and Sin-Itoro Tomonaga (in the case of quantum electrodynamics) around 1950. The renormalizability of the perturbation series, in terms of Feynman integrals, has later been extended to weak interactions and to QCD. The point here is that the external field problem is essentially the only situation for realistic particle physics models where in principle the solution

should be written down in a nonperturbative way. By external field problem I mean here the quantization of the Dirac field but keeping the gauge field classical; in perturbation series only the Feynman propagator for the fermion field appears.

Even in the external field problem there are diverging (1-loop) Feynman diagrams, and these contribute to the phase of the scattering operator. The phase is not purely an academic question since it leads to a modification of the effective action and this in turn modifies the gauge field propagator, which in the case of QED leads to the experimentally very precisely measured modification of certain hydrogen atom energy levels (Lamb shift, [LR]). However, the 1-particle scattering operator S satisfies the technical condition (see below) needed so that the operator can be promoted to an unitary operator \hat{S} in the fermionic Fock space.

In physics the quantum scattering operator is computed using the Feynman rules, the matrix elements are (nonconvergent) sums in the perturbation series, the individual terms given by Feynman diagrams. What is worse, some of the Feynman integrals are diverging, one has to introduce some renormalization methods to subtract the divergent parts. But for the above problem this is not very satisfactory situation since in principle the scattering operator \hat{S} should be well-defined, convergent.

Denoting $U^I(t) = U_0(-t)U(t)$ one could try to determine the phase by a parallel transport, provided that $U^I(t)$ is for all times t in a suitable infinite-dimensional group. One candidate for such a group is $U_{res}(H_+ \oplus H_-)$ for a polarized Hilbert space H . Here H is the Hilbert space of square-integrable fermion fields and the polarization is defined by the sign of the free hamiltonian D_0 . This is natural idea since the operator S is actually in U_{res} . However, for finite times t the time evolution $U^I(t)$ is not in U_{res} .

The phase problem has earlier been discussed from different points of view, [SchFi], [Sch], [GbV], [DDMS], [LM], [Mi98]. A comparison of the different approaches is given in [Laz]. The aim of the present paper is to remove a defect in the earlier work [LM] which is due to an arbitrary choice of a regularization of the 1-particle time evolution for the intermediate times, in order to bring the time evolution operators to U_{res} in the geometric approach proposed in [LM]. The different choices lead to different phases of the scattering operator, the change being given by a holonomy along a closed loop in U_{res} .

In this paper I want to show that there is a natural way to define the parallel transport on an appropriate space of unitary operators (which includes U_{res}) which makes the result independent of the choice of the regularization. The essential ingredient is the splitting of the operators in the parallel transport formula to trace class operators and operators which are conjugate to pseudodifferential operators. Then one can apply a generalized trace calculus to these operators which, when applied to the Feynman diagrams, would be equivalent to the dimensional regularization; in particular, the process eliminates the logarithmic divergencies by subtracting an infinite quantity related to the Guillemin - Wodzicki residue of the operator.

The plan of the paper is the following. In Section 2 the basic geometrical properties of the restricted unitary group U_{res} are recalled. The bulk of the paper is the Section 3 containing the main result (Theorem 1) which shows that there exist a connection defining the parallel transport on a space of unitary operators containing the time evolution operators; this connection defines the phase of the quantum scattering operator. In section 4 the geometric phase is compared with the 1-loop perturbation theory and seen to agree with the dimensional regularisation.

Writing this paper was inspired by many discussions with Dirk Deckert, Detlef Dürr, Franz Merkl, and Martin Schottenloher, and later with José Gracia-Bondía and Joseph Várilly; I want to thank for the invitation to visit LMU in München in September 2009 and University of Zaragoza in October 2010. The completion of the paper was unfortunately delayed for a long time because of other duties and interests.

2 Parallel transport on the group U_{res}

Let \hat{G} be a central extension of a Lie group G by \mathbb{C}^\times . The Lie algebra $\hat{\mathfrak{g}}$ of \hat{G} is a vector space direct sum $\mathfrak{g} \oplus \mathbb{C}$. Let π be the projection on the second summand and let $\theta = dgg^{-1}$ be the right invariant Maurer-Cartan one-form. We can then define a complex valued one-form ϕ on \hat{G} by $\phi = \pi(\theta)$. This is a connection form in the principal \mathbb{C}^\times bundle $\hat{G} \rightarrow G$. Its curvature is a left invariant two-form on G given by $\omega(X, Y) = c(X, Y)$, where left invariant vector fields X, Y on G are identified as elements of the Lie algebra and c is the 2-cocycle on \mathfrak{g} defining the central extension,

$$[(X, \lambda), (Y, \mu)] = ([X, Y], c(X, Y)).$$

Let GL_{res} be the group of invertible linear transformations $g : H \rightarrow H$ such that $[\epsilon, g]$ is Hilbert-Schmidt so that U_{res} is its unitary subgroup. Let us apply the above remarks to $G = GL_{res}$, and to the Lie algebra cocycle c arising when promoting the one-particle operators to operators in the fermionic Fock space.

The central extension \widehat{GL}_{res} is a nontrivial \mathbb{C}^\times bundle over the base GL_{res} , [PrSe]. The elements of the group \widehat{GL}_{res} (containing the unitary subgroup \hat{U}_{res}) can be thought of equivalence classes of pairs (g, q) , where $g \in GL_{res}$ and $q : H_+ \rightarrow H_+$ is an invertible operator such that $a - q$ is a trace-class operator,

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

We have assumed that $\text{ind } a = 0$. If this is not the case, the subspace H_+ must be either enlarged or made smaller by a suitable finite-dimensional subspace in order to achieve $\text{ind } a = 0$. The equivalence relation is determined by $(g, q) \sim (g', q')$ if $g = g'$ and $\det(q'q^{-1}) = 1$. Thus the fiber of the extension is \mathbb{C}^\times and it is parameterized by (the nonexisting) determinant of q . Here we can restrict to the Fredholm index $\text{ind } a = 0$ subgroup since the continuous time evolution starting from the identity operator implies that $U(t)$ for all t is in the connected component of the identity.

The product is defined simply $(g, q)(g', q') = (gg', qq')$. Near the unit element in GL_{res} we can define a local section $g \mapsto (g, a)$, [PrSe]. Denoting

$$g^{-1} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

we can write the connection form as

$$\phi_g = \text{tr} [(dgg^{-1})_a - dq q^{-1}] = \text{tr} [da\alpha + db\gamma - dq q^{-1}]. \quad (1)$$

The curvature of this connection at $g = 1$ is

$$\omega = -\text{tr} (dbdc).$$

Interpreting the tangent vectors at $g = 1$ as elements in the Lie algebra we obtain

$$\omega(X, Y) = -\text{tr} (b(X)c(Y) - b(Y)c(X)) = \frac{1}{4} \text{tr} \epsilon[\epsilon, X][\epsilon, Y].$$

Thus the curvature of the connection is directly given through the Lie algebra central extension as promised.

We compute the parallel transport determined by the connection in the range of the local section. Let $g(t)$ be a path in GL_{res} , $0 \leq t \leq T$, with $g(0) = 1$. The lift $(g(t), q(t))$ is parallel if

$$0 = \phi_{g(t), q(t)}(dg, dq) = \text{tr}[a'(t)\alpha(t) + b'(t)\gamma(t) - q'(t)q(t)^{-1}]. \quad (2)$$

Thus the parallel transport, relative to the trivialization $g \mapsto (g, a)$, along the path $g(t)$ in the base is accompanied with the multiplication by the complex number

$$\exp\left\{-\int_0^T \text{tr}[a'(t)(\alpha(t) - a(t)^{-1}) + b'(t)\gamma(t)]dt\right\} \quad (3)$$

in the fiber \mathbb{C} .

Formally,

$$\text{tr } q'q^{-1} = \text{tr}[a'\alpha + b'\gamma]$$

and so

$$\det q(T) = \exp \int_0^T \text{tr}[a'(t)\alpha(t) + b'(t)\gamma(t)]dt$$

and also

$$\det a(T) = \exp \int_0^T \text{tr } a'(t)a(t)^{-1}dt.$$

Individually, the traces in these two expressions do not converge, but putted together the trace converges and gives

$$\det(a(T)q(T)^{-1}) = \exp\left\{\int_0^T \text{tr}[(a'(\alpha - a^{-1}) + b'\gamma)]dt\right\}. \quad (4)$$

Note that the exponent diverges outside of the domain of the local section, reflecting the fact that $\det a(T) = 0$ outside of the domain.

3 Time evolution of fermions in external gauge fields

We shall study massless Dirac fermions coupled to a gauge potential A in Minkowski space. The potential is a smooth 1-form $A_\mu(x) dx^\mu$ in space-time with values in the Lie algebra \mathfrak{g} of a compact gauge group G . The elements of \mathfrak{g} are represented by hermitean (according to physics literature convention) matrices in the complex vector space \mathbb{C}^N . The free Dirac operator is then $i \sum_{\mu=0}^d \gamma^\mu \partial_\mu + m$. The metric is $x^2 = g_{\mu\nu}x^\mu x^\nu = x_0^2 - x_1^2 - \dots - x_d^2$. The Dirac gamma matrices satisfy $\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2g_{\mu\nu}$, γ_0 is hermitean and γ_k is antihermitean for $k \neq 0$. The Dirac hamiltonian in the background gauge field A is

$$D_A = -\gamma^0 \sum_{k=1}^d \gamma^k (i\partial_k + A_k) - A_0 - \gamma_0 m,$$

where $m \geq 0$ is the mass of the fermion. We shall take as the initial condition, at $t = 0$, $A(\mathbf{x}, t = 0) = 0$. We shall assume that $A(x)$ and its derivatives vanish faster than $|x|^{-d/2}$ when $|x| \rightarrow \infty$.

The unitary time evolution operator of Dirac fermions in an external gauge potential A is given as the solution of

$$i\partial_t U(t) = (D_0 + V)U(t)$$

with the initial condition $U(0) = 1$ where $D_0 = -i\gamma^0\gamma^k\partial_k - \gamma_0m$ and $V = -i\gamma^0\gamma^k A_k - A_0$ is the interaction term. Even in the case of QED the time evolution $U(t)$ is not in the group U_{res} , except in the case of pure electric field [Rui], [Pal]. According to [Mi94], [LM], we can choose an unitary operator T_A which depends on A and its time and space derivatives such that the renormalized time evolution $U_{ren}(t) = T_A U(t)$ satisfies the differential equation

$$i\partial_t U_{ren}(t) = (D_0 + V_{ren})U_{ren}(t),$$

where

$$V_{ren} = T_A^{-1} V T_A + T_A^{-1} i\partial_t T_A$$

is such that the commutator $[\epsilon, V_{ren}] \in L_2$. Here $\epsilon = \epsilon(D_0)$ is defined through the spectral step function $\epsilon(x) = -1$ for $x < 0$ and $\epsilon(x) = +1$ for $x \geq 0$. Actually, it is possible to choose T_A such that the commutator is even a trace-class operator [LM]. The operator T_A is a pseudodifferential operator which differs from the unit operator by a pseudodifferential operator of order -1 .

Going to the interaction picture, $U_{ren}^I = e^{itD_0} U_{ren}$, we have

$$i\partial_t U_{ren}^I(t) = V_{ren}^I U_{ren}^I(t)$$

with $V_{ren}^I = e^{itD_0} V_{ren} e^{-itD_0}$. Now the interaction V_{ren}^I is an element of the Lie algebra \mathfrak{u}_1 of the group $U_1(H)$ consisting of unitaries in $H = H_+ \oplus H_-$ with trace-class off-diagonal blocks and is a continuous function of the time t . It follows that the time evolution equation has a differentiable (in time) solution $U_{ren}^I(t)$.

We can now apply the above results to the 'renormalized' one-particle time evolution operators $g(t) = U_{ren}^I(t)$ in the interaction picture. Let us, for the sake of simplicity, assume that the interaction is switched off outside of a finite interval $[0, T]$ in time. Thus the 1-particle scattering operator is $S_A = U_{ren}^I(T) = U(T)$. For all times t , $g(t) \in U_1$. On the other hand, in the Fock representation of \widehat{GL}_1 these correspond to elements $\hat{g}(t)$ in the central extension \hat{U}_1 . The phase of the quantum time evolution operator is then uniquely given by the parallel transport described above.

The Minkowskian effective action $Z(A)$ is by definition the vacuum expectation value of the quantum scattering operator \hat{S}_A . The vacuum is invariant under the free time evolution $\exp(itD_0)$ and taking into account the assumption that the interaction has essentially compact support in time, we can write

$$Z(A) = \langle 0 | (g(T), q(T)) | 0 \rangle.$$

The vacuum expectation value is given by a simple formula, [LM],

$$\langle 0 | (g, q) | 0 \rangle = \det(aq^{-1})$$

and therefore the parallel transport (with respect to the given local trivialization) defines the phase of the effective action $Z(A)$.

Actually, the above discussion can be extended by a slight modification to the non-renormalized time evolution operators $U^I(t)$ in the interaction picture. This is important since there is a great freedom in the choice of the family of unitary operators T_A ; basically $\tilde{T}_A = T_A Q_A$ is acceptable renormalization for any $Q_A \in U_{res}$. We want to have a formula which does not depend on the choice of T_A .

Lemma 1 *The time evolution in the interaction picture can be factorized as*

$$U^I(t) = (e^{itD_0} T_A^{-1} e^{-itD_0}) U_{ren}^I(t)$$

such that the off-diagonal blocks of $U_{ren}^I(t)$ are trace-class operators and $T_A - 1$ is a pseudodifferential operator of order -1 .

Proof Denote by H the Hilbert space of square integrable spinor fields on \mathbb{R}^d , D_0 the free Dirac operator, and the grading $\epsilon = D_0/|D_0|$.

The time evolution of Dirac fermions in an external gauge potential A is given as

$$i\partial_t U(t) = (D_0 + V)U(t)$$

where $V = \alpha_k A_k$ is the interaction term. According to [LM], we can choose an unitary operator T_A which depends on A and its time and space derivatives such that the renormalized time evolution $U_{ren}(t) = T_A U(t)$ satisfies the differential equation

$$i\partial_t U_{ren}(t) = (D_0 + V_{ren})U_{ren}(t),$$

where

$$V_{ren} = T_A^{-1} V T_A + T_A^{-1} i\partial_t T_A$$

is such that the commutator $[\epsilon, V_{ren}] \in L_2$. Actually, it is possible to choose T_A such that the commutator is even a trace-class operator, [LM]. The operator T_A is a pseudodifferential operator which differs from the unit operator by a pseudodifferential operator of order -1 .

Going to the interaction picture, $U_{ren}^I = e^{itD_0} U_{ren}$, we have

$$i\partial_t U_{ren}^I(t) = V_{ren}^I U_{ren}^I(t)$$

with $V_{ren}^I = e^{itD_0} V_{ren} e^{-itD_0}$. Now the interaction V_{ren}^I is an element of the Lie algebra \mathfrak{u}_1 of the group U_1 and is a continuous function of the time t . It follows that the time evolution equation has a differentiable (in time) solution $U_{ren}^I(t)$.

We can now factorize the original time evolution $U(t)$ as

$$U^I(t) = e^{itD_0} U(t) = e^{itD_0} T_A^{-1} e^{-itD_0} U_{ren}^I(t).$$

□

We shall use a trace extension on pseudodifferential operators, called the weighted trace by S. Paycha [Pa]. It gives the usual operator trace for trace-class operators, that is, on a compact manifold M for pseudodifferential operators of order strictly less than $-\dim M$. However, it is not cyclic for general pseudodifferential operators. First, one fixes a *weight* as an elliptic invertible operator Q of positive order q . If T is a pseudodifferential operator then the function $f(z) = \text{tr } Q^{-z} T$ is holomorphic in a half-plane $\text{Re}(z) > \frac{1}{q}(\text{ord}(T) + \dim M)$ and can be continued to an analytic function in the neighborhood of $z = 0$ with a simple pole at $z = 0$. The weighted trace of T is defined as

$$\text{tr}_Q T = \lim_{z \rightarrow 0} \left(\text{tr } Q^{-z} T - \frac{1}{qz} \text{Res } T \right)$$

where $\text{Res } T$ is the Guillemin- Wodzicki residue of T . Although the trace is not cyclic, the defect is given by the simple formula

$$\text{tr}_Q [T, S] = -\frac{1}{q} \text{Res } T [\log Q, S].$$

Although the logarithm of Q is not a classical pseudodifferential operator, the commutator $[\log Q, S]$ is since its symbol is composed of the derivatives of the logarithm.

Remark One can define a \mathbb{C}^\times bundle over the space of all bounded invertible operators g in $H = H_+ \oplus H_-$ such that the block a is a Fredholm operator as in the case of the group U_{res} : The total space is the set of pairs (g, q) with $a - q$ trace-class, and the equivalence relation is $(g, q) \sim (g', q')$ for $g = g'$ and $\det(qq'^{-1}) = 1$. However, the total space is not a group.

Theorem 1 *Choosing the local section $q = a_g$ in the connection form $\phi = \text{tr}_Q[(dgg^{-1})_a - dqg^{-1}]$, for $g(t) = U^I(t)$ the operator under the trace is of the form $e^{itD_0}Xe^{-itD_0} + Y$ where X is a pseudodifferential operator of order -1 and Y is a trace-class operator. Thus choosing $Q = |D_0|$ the weighted trace tr_Q is well-defined and equal to $\text{tr}Y + \text{tr}_QX$ since Q commutes with e^{itD_0} . The result does not depend on the choice of splitting (X, Y) .*

Proof Write $g(t) = U^I(t) = g_1(t)g_2(t)$ with $g_1(t) = e^{itD_0}T_A^{-1}e^{-itD_0}$ and $g_2(t) = U_{ren}^I(t)$. Then $(dg(t)g(t)^{-1})_a = V^I(t)_a$ and

$$da = P_+dgP_+ = P_+V^I(t)g(t)P_+ = (P_+V^IP_+)a + (P_+V^IP_-)(P_-gP_+)$$

and therefore

$$daa^{-1} = P_+V^IP_+ + (P_+V^IP_-)(ca^{-1}).$$

The first term on the right-hand-side cancels $(dgg^{-1})_a$ in ϕ whereas the second term is equal to

$$\begin{aligned} & (P_+V^IP_-)(c_1a_2 + d_1c_2)(a_1a_2 + b_1c_2)^{-1} \\ &= (P_+V^IP_-)(c_1a_1^{-1} + d_1c_2a_2^{-1}a_1^{-1})(1 + b_1c_2a_2^{-1}a_1^{-1}) \equiv (P_+V^IP_-)c_1a_1^{-1} \pmod{L_1} \end{aligned}$$

since $c_2 \in L_1$. On the other hand, $c_1 = P_-e^{itD_0}T_A^{-1}e^{-itD_0}P_+$ and $a_1 = P_+(\dots)P_+$. It follows that the product $(P_+V^IP_-)ca^{-1}$ is conjugate (by e^{itD_0}) to a pseudodifferential operator of order -1 , modulo trace-class operators.

Finally, let us assume that $e^{itD_0}Xe^{-itD_0} + Y = e^{itD_0}X'e^{-itD_0} + Y'$ where also Y' is trace-class and X' is a pseudodifferential operator of order -1 . Then $X - X'$ has to be also trace class since L_1 is an ideal in the space of bounded operators. Since tr_Q is the standard trace for trace class operators we see that $\text{tr}_Q e^{itD_0}(X - X')e^{-itD_0} = \text{tr} e^{itD_0}(X - X')e^{-itD_0} = -\text{tr}(Y - Y')$ which implies the uniqueness of the total trace.

□

Remark The above result can be formulated more generally: Let \mathcal{G} be a space consisting of unitary operators $g = g_1g_2$ with g_1 a conjugate of a pseudodifferential operator by a unitary operator R with $RQ = QR$, and $g_2 \in U_{tr}$. Let $t \mapsto g(t)$ be a differentiable path such that $dgg^{-1} = R(t)V(t)R(t)^{-1}$ and $g_1(t) = R(t)g_0(t)R(t)^{-1}$ for pseudodifferential operators $V(t), g_0(t)$. Then the formula (2), with tr replaced by tr_Q , defines a connection in a \mathbb{C}^\times bundle over \mathcal{G} . However, \mathcal{G} is not a group.

4 Comparison with perturbation theory

Together with formula (4) the above Theorem gives a method to compute the effective action $\log(Z)$. Let us consider the case of QED in four space-time dimensions. Our method is nonperturbative, but using the Dyson expansion

$$g(t) = 1 - i \int_{-\infty}^t V_I(s)ds + (-i)^2 \int_{t > s_1 > s_2} V_I(s_1)V_I(s_2)ds_1ds_2 + \dots \quad (5)$$

for the time evolution operator in the interaction picture we get the lowest (A^2 term) for $\log(Z)$,

$$\log(Z) = \int_{s > t} \text{tr} \pi_+ V_I(s) \pi_- V_I(t) \pi_+ dt ds. \quad (6)$$

In the case of QED, the terms of odd order are identically zero by parity invariance and the fourth order term is already finite as a Feynman integral. So actually the second order term is the most interesting.

So let us compute $\log(Z)$ to second order in the interaction A in the case of QED, i.e., massive fermion coupled to a Maxwell potential. Because of the unitarity relation $a^*a + c^*c = 1$ the inverse $a^{-1} \equiv a^*$ modulo terms of order A^2 since the off diagonal blocks of the time evolution $g(t)$ must contain the potential at least to order one. For this reason the term $(\alpha - a^{-1})a'$ in the phase does not give contributions to the order A^2 . On the other hand, because of unitarity, $\beta c' = c^*c' \equiv -bc'$ modulo terms of order higher than two.

We shall use the integral representation

$$\begin{aligned} & \frac{1}{2\pi i} \int \text{tr} \frac{\not{p} - m}{p^2 - m^2 + i\epsilon} e^{ip_0 T} dp_0 \\ &= \gamma_0 [\theta(T)\theta(h_0(\mathbf{p}))e^{-iT\omega_p} - \theta(-T)\theta(-h_0(\mathbf{p}))e^{iT\omega_p}] \end{aligned} \quad (7)$$

where $\omega_p = \sqrt{\mathbf{p}^2 + m^2}$, $\theta(x) = 1$ for $x \geq 0$ and $\theta(x) = 0$ for $x < 0$ and $h_0(\mathbf{p}) = \gamma^0 \gamma^k p_k + \gamma^0 m$ is the momentum representation for the free Dirac hamiltonian. In the usual QED perturbation theory the second order effect is given by the (diverging) Feynman integral

$$\frac{1}{4\pi} \int \text{tr} \frac{\not{p} - m}{p^2 - m^2 + i\epsilon} \not{A}(p - q) \frac{\not{q} - m}{q^2 - m^2 + i\epsilon} \not{A}(q - p) d^4 p d^4 q \quad (8)$$

with the Dirac notation $\not{X} = \sum \gamma_\mu X^\mu$. By a Fourier transform of the potential in the time variable this integral can be written as

$$\frac{1}{8\pi^2} \int \text{tr} \frac{\not{p} - m}{p^2 - m^2 + i\epsilon} e^{is(p_0 - q_0)} \not{A}(s, \mathbf{p} - \mathbf{q}) \frac{\not{q} - m}{q^2 - m^2 + i\epsilon} e^{it(q_0 - p_0)} \not{A}(t, \mathbf{q} - \mathbf{p}) ds, dt d^4 p d^4 q.$$

Using the trick (7) above we can write the integral as

$$\begin{aligned} & -\frac{1}{2} \int \text{tr} [\theta(s - t)\theta(h_0(\mathbf{p}))e^{-i(s-t)h_0(\mathbf{p})} - \theta(t - s)\theta(-h_0(\mathbf{p}))e^{-ih_0(\mathbf{p})(s-t)}] \\ & \times \not{A}(s, \mathbf{p} - \mathbf{q}) [\theta(t - s)\theta(h_0(\mathbf{q}))e^{-i(t-s)h_0(\mathbf{q})} - \theta(s - t)\theta(-h_0(\mathbf{q}))e^{-i(t-s)h_0(\mathbf{q})}] \\ & \times \not{A}(t, \mathbf{q} - \mathbf{p}) e^{-ih_0(\mathbf{q})(t-s)} ds dt d^3 \mathbf{p} d^3 \mathbf{q}. \end{aligned} \quad (9)$$

Since $\theta(T)\theta(-T) = 0$ and $\theta(T)^2 = \theta(T)$ we get

$$\int \text{tr} \theta(s - t) \pi_+ V_I(t) \pi_- V_I(s) \pi_+ ds dt$$

which is exactly the second order term in our geometric definition of $\log(Z)$. Note that this discussion is formal in the sense that diverging Feynman integrals are involved. However, we may apply some renormalization method (for example, the family of T_A operators described earlier, to bring the time evolution to the group U_1) in order to make sense of these integrals. But using our definition of parallel transport in terms of the weighted trace $\text{tr}_{|D|}$ means that we are removing the logarithmic divergence in the 1-loop diagram, which in the dimensional regularisation is the term $\frac{1}{z} \text{Res}$. The limit $z \rightarrow 0$ is just the limit $\epsilon \rightarrow 0$ in the dimension $4 + \epsilon$.

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